DESIGN OF A STOCHASTIC LINEAR DISCRETE REGULATOR BASED ON AN OPTIMAL STATE ESTIMATION APPROACH

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Abstract – The problem of regulator design for stochastic linear discrete-time systems by optimizing a quadratic index of performance is considered. Loose separability is assumed and one-step-ahead minimization of control magnitude and of controlled state deviation from zero is imposed to obtain the control law. To determine the weighting matrices in the index of performance, a state estimation based scheme of analysis is proposed. With this approach these matrices are taken as covariance matrices, through the use, one step ahead, of the information about allowed control magnitude and possible and desirable state deviation from zero. Convergence of the proposed regulator is analysed also using a state estimation equivalent problem approach. Complete observability and complete controllability are shown to guarantee the existence of a sequence of stabilizing control values, which can be estimated by the adopted optimization design approach. The structure of the resulting control law depends only on past and present knowledge of system dynamics, making the proposed solution suitable for applications where adaptive control is necessary. Results obtained in one case of satellite attitude control are discussed.

1 – INTRODUCTION

In the usual LQG solution, the resulting controller depends on knowledge of future system dynamics to be implemented (e.g. Bryson and Ho, 1969). This is a serious limitation when system modelled dynamics is only a local approximation and adaptive control schemes are necessary.

In this paper, the design of a stochastic linear regulator is proposed for the case of linear discrete-time systems according to a strategy that depends only on past and present information about the dynamic system. This is done by (i) using an optimization scheme as in the LQG solution, but adopting an index of performance that only includes one-step-ahead minimization of control and state deviations from zero; and (ii) looking at the weighting matrices present in the index of performance as covariance matrices, and using a state estimation based scheme of analysis to determine these matrices.

To analyse convergence of the proposed solution, the problem of existence and estimation of a sequence of stabilizing control values is also posed as an equivalent estimation problem. It is the shown that: the properties of observability and controllability of the original control problem imply the existence of a stabilizing sequence; and that this sequence can be estimated by the proposed optimization scheme of design.

As expected, the expression of the control law gain is similar to that of the LQG solution. The only difference is in the way that the state weighting matrix is defined. As a consequence of this, one gets a regulator for which complete separability (e.g. Jacobs, 1981) does not hold, since the control gain results dependent on state uncertainty.

The results presented here come after some previous heuristic efforts to use a similar strategy in the design of adaptive controllers applied to ship and satellite control (Rios-Neto and Cruz, 1985; Ferreira, Rios-Neto and Venkataraman, 1985). As a matter of fact they were developed looking for the establishment of a theoretical basis for the heuristic results used in those applications.

In the sections ahead the paper is organized as follows. In section 2, the problem of controlling a linear discrete dynamic system is formulated to introduce the notation and to establish the basic assumptions. In section 3, the proposed solution is presented. In section 4, it is analysed from the point of view of convergence. Test results obtained in one case of satellite attitude control is discussed in section 5. Finally, in section 6 the conclusions are presented.

2 – PROBLEM FORMULATION

The problem is that of controlling a linear discrete-time dynamic system for which noise corrupted observations are given at each sample time, i = 0, 1, 2, ...:

$$x(i+1) = \phi(i+1)\,x(i) + \Gamma(i)\,u(i) + w(i)$$

$$y(i) = H(i)x(i) + v(i), \tag{1}$$

where x(i) is the nx1 state vector; u(i) the rx1 control vector; y(i) the mx1 observation vector; and v(i), w(i), x(0) are zero mean Gaussian vectors, with compatible dimensions, such that for i,j = 0, 1, 2, ...:

$$E[v(i)w^t(j)] = 0,\; E[x(0)w^t(i)] =$$
$$= 0,\; E[x(0)v^t(i)] = 0,$$
$$E[x(0)x^t(0)] = \bar{P}(0) > 0,\; E[v(i)v^t(j)] =$$
$$= R(i)\,\delta ij,$$
$$E[w(i)w^t(j)] = Q(i)\,\delta ij,$$

where $E[.]$ indicates the expected value of its argument, and δij is the delta of Kronecker.

The system of Equations (1) is assumed to be completely observable and completely controllable. The objective is to control the state x(i) towards zero as time increases. That is, one looks for a control strategy that stabilizes the system, leading to a regulator type solution.

To realize this objective, in the next section a procedure will be proposed using a linear state feedback control law, where only a loose separation (e.g. Jacobs, 1981) holds. It will then be necessary to define a controlled state as:

$$\bar{x}(i+1) = (\phi(i+1,i) - \Gamma(i)C(i))\,\hat{x}(i)$$
$$= (\phi(i+1,i) - \Gamma(i)\,C(i))\,(\bar{x}(i) -$$
$$- K(i)\,(H(i)\bar{e}(i) - v(i))), \tag{2}$$

where $\hat{x}(i)$ is the Kalman filtering state estimate after measurement y(i) is processed (e.g. Jazwinski, 1970); K(i) is the Kalman gain; C(i) is the linear state feedback control law gain; $\bar{e}(i) \triangleq \bar{x}(i) - x(i)$; $\bar{x}(0) = N(0; \bar{X}(0))$. The controlled states constitute a stochastic zero mean Gaussian sequence with second order moments given by (Bryson and Ho, 1969):

$$\bar{X}(i+1) = (\phi(i+1,i) - \Gamma(i)C(i))(\bar{X}(i) + \bar{P}(i) -$$
$$- P(i))(\phi(i+1,i) -$$
$$\Gamma(i)C(i))^t, \tag{3}$$

where $\bar{X}(i) \triangleq E[\bar{x}(i)\bar{x}^t(i)]$; $\bar{P}(i) \triangleq E[\bar{e}(i)\bar{e}^t(i)]$; $P(i) \triangleq E[(\hat{x}(i) - x(i))(\hat{x}(i) - x(i))^t]$; and $C(-1) = 0$, $\bar{X}(0) = \bar{P}(0)$.

3 – PROPOSED SCHEME

To determine a regulator type of control law for the problem of Equations (2), one ends up solving a deterministic optimization problem in the proposed scheme. At each time t_i, $u(i)$ is determined as the value that minimizes the index of performance:

$$J = 1/2(u^t(i)B(i)u(i) +$$
$$(\bar{x}^{pd}(i+1) - \bar{x}(i+1))^t S(i+1)(\bar{x}^{pd}(i+1) - \bar{x}(i+1))) \tag{4}$$

subject to the dynamic constraint

$$\bar{x}(i+1) = \phi(i+1,i)\hat{x}(i) + \Gamma(i)u(i), \tag{5}$$

where $\hat{x}(i)$ is known and given by the realization coming out of the Kalman filter state estimate after processing the observed value of y(i), at t_i; $B(i) > 0$ is given and chosen according to the bounds within which u(i) is constrained; $\bar{x}^{pd}(i+1)$ is given and previously determined to guide the predicted state towards a desired contraction of state magnitude at t_{i+1}; and $S(i+1) > 0$ is given and previously determined so as to condition a realization of $(\bar{x}^{pd}(i+1) - \bar{x}(i+1))$ that is within bounds that are both possible and desirable in terms of the objective of the regulator being designed.

The determination of $S(i+1)$ and $\bar{x}^{pd}(i+1)$ is done in such a way as to include and mix two types of information. In a sense that will be made clear in what follows, one first considers a predicted and possible distribution for $\bar{x}(i+1)$ and, second, imposes a desired contraction on the state magnitude, at t_{i+1}. This can be done as shown in the following three steps:

First step: A possible and predicted controlled state can be defined as being virtually given by:

$$\bar{x}^{p}(i+1) = \phi(i+1, i)\hat{x}(i) - \Gamma(i)C(i-1)\hat{x}(i) \tag{6}$$

This is the controlled state that results when the matrix C(i) of the control in Equation (2) is considered with one sample interval lag. Thus, $\bar{x}^{p}(i+1)$ is certainly a possible controlled state that can be reached. It can be viewed not as a particular realization by as a random variable out of a stochastic process, which from Equation (1) and the properties of $\hat{x}(i)$ is certainly zero mean and Gaussian (Bryson and Ho, 1969). In this sense, a possible predicted dispersion for $\bar{x}(i+1)$ results:

$$\bar{X}^{p}(i+1) = (\phi(i+1, i) - \Gamma(i)C(i-1))(\bar{X}(i) + \bar{P}(i) - P(i)) (\phi(i+1, i) - \Gamma(i)C(i-1)) \tag{7}$$

where $\bar{X}^{p}(i+1) \triangleq E[\bar{x}^{p}(i+1)\bar{x}^{p^{t}}(i+1)]$ and the other variables are as already defined in Equation (3). One can then look for a strategy that leads to an occurred value of $\bar{x}(i+1)$ as a convenient outcome of the Gaussian random variable $\bar{x}^{p}(i+1)$, where:

$$\bar{x}^{p}(i+1) = N(0; \bar{X}^{p}(i+1)) \tag{8}$$

Second step: A desired response at t_{i+1} for the system under control can be viewed and defined as those realizations of $\bar{x}^{p}(i+1)$ which are sufficiently close to zero to guarantee the objective of controlled state magnitude contraction. In a virtual sense, it is possible to consider a sensor with the capacity of directly observing the state $\bar{x}^{p}(i+1)$. Still in a virtual sense, it is also possible to imagine that one is observing, with an imposed accuracy, a realization of $\bar{x}^{p}(i+1)$ that satisfies the condition of being a desired response. This situation can be formally expressed by:

$$\bar{y}^{d}(i+1) = \bar{x}^{p}(i+1) + \bar{v}^{d}(i+1) \tag{9}$$

where $\bar{y}^{d}(i+1)$ is chosen to constitute the virtually observed desired response, at t_{i+1}; $\bar{v}^{d}(i+1) = N(0, \bar{R}^{d}(i+1))$, with $\bar{R}^{d}(i+1)$ assumed diagonal, with variances properly chosen to characterize virtual sensor errors. The accuracy of the sensors is chosen such as to guarantee that, with a very high probability, the $\bar{x}^{p}(i+1)$ in correspondence with $\bar{y}^{d}(i+1)$ is within the region of a desired response.

Third step: Combining the a priori information of Equation (8) with the observation of Equation (9) one can apply an optimal Gauss-Markov linear estimator (e.g. Jazwinski, 1970) to obtain an estimate, $\hat{\bar{x}}^{p}(i+1)$, of a possible and desirable state, among those attainable at t_{i+1}. Together with $\hat{\bar{x}}^{p}(i+1)$ one also obtains the covariance matrix $P^{p}(i+1)$ of the errors $(\hat{\bar{x}}^{p}(i+1) - \bar{x}^{p}(i+1))$. The second term of the Performance Index (4) is then completely defined if one chooses:

$$\bar{x}^{pd}(i+1) \triangleq \hat{\bar{x}}^{p}(i+1), \; S(i+1) = (P^{p}(i+1))^{-1} \tag{10}$$

where in the first equation $\bar{x}^{pd}(i+1)$ is to be understood as a realization of $\hat{\bar{x}}^{p}(i+1)$ that results when a chosen $\bar{y}^{d}(i+1)$ sufficiently close to zero is processed; and $S(i+1)$, in the second equation, is in correspondence with a chosen $\bar{R}^{d}(i+1)$ that leads to $(\hat{\bar{x}}^{p}(i+1) - \bar{y}^{d}(i+1))$ within bounds that guarantee the objective of controlled state contraction at t_{i+1}. From the expressions of the linear estimator, $S(i+1)$ results as:

$$S(i+1) = (\bar{R}^{d}(i+1))^{-1} + (\bar{X}^{p}(i+1))^{-1} \tag{11}$$

For the particular situation where the $\bar{y}^{d}(i+1)$ is chosen to be zero, it results:

$$\bar{x}^{pd}(i+1) = \hat{\bar{x}}^{p}(i+1) = 0. \tag{12}$$

For this particular situation, the controller that results from the minimization of the Index (4), subject to the constraint of Equation (5), is as follows:

$$u(i) = -C(i)\hat{x}(i) \tag{13}$$

$$C(i) = (\Gamma^{t}(i)S(i+1)\Gamma(i) + B(i))^{-1} \Gamma^{t}(i)S(i+1)\phi(i+1, i)$$
$$= B^{-1}(i)\Gamma^{t}(i)(\Gamma(i)B^{-1}(i)\Gamma^{t}(i) + S^{-1}(i+1))^{-1} \phi(i+1, i) \tag{14}$$

where, except for the meaning given to $S(i+1)$, the expression for C(i) is the same as that for the optimal regulator (e.g. Bryson and Ho, 1969).

2×3,14 — 360 $x = \left(\frac{2\times 3,14}{360}\right)\times 0,05$
x — 0,05

4 – CONVERGENCE ANALYSIS

By construction, consider the following virtual estimation problem, which is related to but not to be confused with that of Equations (1):

$$\bar{x}(i+1) = \phi(i+1,i)\bar{x}(i) + \Gamma(i)u(i) + \bar{w}(i) \qquad (15)$$

$$\bar{y}(i+1) = \bar{H}(i+1)\bar{x}(i+1) + \bar{v}(i+1) \qquad (16)$$

where $\bar{x}(i+1)$ is the controlled state (see Equation (5), substituting the expression for $\hat{x}(i)$, as given by the Kalman filter when it is applied to problem of Equation (1); $\bar{w}(i)$ is such that $\phi(i+1,i)\hat{x}(i) = \phi(i+1,i)\bar{x}(i) + \bar{w}(i)$ and which from the observability of System (1) converges to a white noise having the same distribution as $\phi(i+1,i)K(i)v(i)$ (Kailath,1968), when the Kalman filter is applied to the problem of Equations (1); $\bar{H}(i+1) \triangleq I_n$; and $\bar{v}(i+1) \triangleq N(0, S^{-1}(i+1))$, a white noise taken as independent of $\bar{w}(i)$, $u(i)$ and $\bar{x}(i+1)$, by hypothesis and with $S(i+1)$ as given by Equation (11).

The estimation problem of Equations (15) and (16) is certainly observable, because $\bar{H}(i+1) \triangleq I_n$. If, at t_i, $u(i)$ and $\bar{w}(i)$ are considered as being the outcomes of Gaussian white noise sequences, with $u(i) = N(0, B^{-1}(i))$, and $\bar{w}(i) = N(0, \phi(i+1,i)K(i)R(i)\phi^T(i+1,i)K^T(i))$ (with dispersion usually negligible as compared to that of $u(i)$), then from the controllability $(u(i))$, and observability $(\bar{w}(i))$ of the original problem of Equations (1) there results the controllability of the estimation problem of Equations (15), (16). Thus convergence is guaranteed for this estimation problem (e.g. Jazwinski, 1970), implying that the estimate $\hat{\bar{x}}(i+1)$, given by the Kalman filter, leads to a residue:

$$\bar{r}(i+1) = \bar{y}(i+1) - \hat{\bar{x}}(i+1) \qquad (17)$$

that necessarily converges to the associated innovation process, reaching in the limit, the same distribution as that of $\bar{v}(i+1)$. This behavior of the residue $\bar{r}(i+1)$ means that the objective of controlling the state of System (1), to make it to approach zero, can be reached. The convergence guarantees the existence of a sequence of occurred values of $u(i)$ that, together with the correspondent occurred values of the noises $\bar{w}(i)$ and $\bar{v}(i+1)$, reproduces the observed $\bar{y}(i+1)$. If the observed realizations are forced to be the certainly possible outcomes $\bar{y}(i+1) = \bar{x}^{pd}(i+1)$, this sequence of u(i) satisfies the control objective posed for the system of Equation (1). Since $\hat{x}(i)$ is known and considering the uncertainties imposed by $\bar{w}(i+1)$ and $\bar{v}(i+1)$ the best way to recover this sequence of the controls $u(i)$ is to solve the following parameter estimation problem:

$$0 = u(i) + \mathcal{E}_u(i)$$

$$\bar{y}(i+1) - \phi(i+1,i)\hat{x}(i) = \Gamma(i)u(i) + \bar{v}(i+1) \qquad (18)$$

where $\mathcal{E}_u(i) \triangleq N(0, B^{-1}(i))$; and the second equation resulted from the substitution of $\bar{x}(i+1)$, as given by Equation (5) (or, equivalently, by Equation (15)), in Equation (16).

The solution of the estimation problem of Equation (18), using a Gauss Markov optimal estimator, is formally equivalent to that resulting from the minimization of the Index of Performance (4) subject to the constraint of Equation (5). Thus, the proposed controller should lead to the stabilization of system of Equation (1).

5 – NUMERICAL TEST: BIAS MOMENTUM ATTITUDE CONTROL SYSTEM

With the objective of illustrating the convergence behavior of the proposed approach one example is numerically simulated in what follows. It is related to satellite three axis attitude control.

This case is based upon a model of double-gimbaled momentum wheel given by Kaplan (1976) for the attitude control of a geostationary satellite. In discrete-time form the state model is as follows:

$$\phi(i+1,i) = \begin{bmatrix} 1 & 0 & 1{,}21E{-}10 & 0{,}1 & 0 & -5{,}0E{-}04 \\ 0 & 1 & 0 & 0 & 0{,}1 & 0 \\ -1{,}21E{-}10 & 0 & 1 & 5{,}0E{-}04 & 0 & 0{,}1 \\ -7{,}28E{-}07 & 0 & 3{,}64E{-}09 & 1 & 0 & -0{,}01 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -3{,}64E{-}09 & 0 & -7{,}28E{-}07 & 0{,}01 & 0 & 1 \end{bmatrix}$$

$$\Gamma(i) = \begin{bmatrix} 0{,}25E{-}05 & 0 & -8{,}33E{-}09 \\ 0 & 1{,}25E{-}05 & 0 \\ 8{,}33E{-}09 & 0 & 2{,}5E{-}07 \\ 5{,}0E{-}05 & 0 & -2{,}5E{-}07 \\ 0 & 2{,}5E{-}04 & 0 \\ 2{,}5E{-}07 & 0 & 5{,}0E{-}05 \end{bmatrix} \qquad (19)$$

$$H(i) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R(i) = diag(3{,}4E{-}09 \; ; \; 3{,}4E{-}09)$$

$$Q(i) = diag(1{,}73E{-}29 \; ; \; 1{,}09E{-}24 \; ; \; 1{,}75E{-}29 \; ; \; 6{,}90E{-}27 \; ; \; 4{,}34E{-}22 \; ; \; 7{,}01E{-}27)$$

where the state vector components are the roll, pitch and yaw angles and their time rates.

The model above was derived based on a discretization time interval of 0,1 s for a satellite with the following parameters (Kaplan, 1976):

Moments of inertia:

$: I_x = I_z = 2000 \ N.m.s^2$

$I_y = 400 \ N.m.s^2$

Nominal Wheel Momentum:

$h_n = 200 \ N.m.s$

Orbital Frequency:

$: w_o = 7{,}28E-05 \ rad/s$

The satellite axes x, y and z are respectively in correspondence with roll, pitch and yaw. The wheel axis coincides with the y axis.

The controller parameters and initial conditions were as follows:

$$C(-1) = 0$$

$$\bar{X}(0) = \dot{\bar{P}}(0) = diag\ (3{,}4E-05\ ;\ 3{,}4E-05\ ;\ 3{,}4E-05\ ;\ 3{,}4E-07\ ;\ 3{,}4E-07\ ;\ 3{,}4E-07\ )$$

$$B(i) = diag\ (1\ ;\ 10\ ;\ 1) \tag{20}$$

$$\bar{R}^d(i+1) = diag\ (\beta_j(i+1),\ j=1,2,\dots\dots\dots,6)$$

where:

$$\beta_j(i+1) = \begin{cases} \alpha_j \bar{x}_j^P(i+1)^2 & \text{if } \alpha_j \bar{x}_j^P(i+1)^2 \geq \&_j \\ \&_j & \text{if } \alpha_j \bar{x}_j^P(i+1)^2 < \&_j \end{cases} \tag{21}$$

and:

$$\alpha_1 = \alpha_2 = \alpha_3 = 0{,}01$$

$$\alpha_4 = \alpha_5 = \alpha_6 = 1$$

$$\&_1 = \&_2 = 8{,}5E-08 \tag{22}$$

$$\&_3 = 5{,}0E-06$$

$$\&_4 = \&_5 = \&_6 = 3{,}4E-07$$

In order to avoid near singularity related problems, the diagonal elements of the matrix $\bar{X}P(i+1)$ were saturated from below at levels $e_j\ (j=1,2,\dots,6)$.

The simulation results are shown in figure 1.

For the kind of application at hand and when comparedt to the controller (based on classic frequency domain techniques) used by Kaplan (1976), the controller exhibits a faster response when subjected to perturbations in the initial attitude angles, reaching afterwards a condition of satisfactory error levels. The worse results in yaw were expected since this state is not directly observed (see H(i) in Equation 14)).

In addition, tests were also carried out with a satellite attitude control problem where only one thruster was available for three-axis control. This was done using a model given by Muller and Weber (1972). Results obtained in this case were still satisfactory despite the fact of having a single actuator, which characterizes an adverse situation.

6 – CONCLUSIONS

A regulator for stochastic discrete-time linear systems has been proposed. It is a sequential state estimate feedback type of controller. Its main feature is to have a control gain matrix dependent only on past and present knowledge about system dynamics and state estimate uncertainties.

The fact that knowledge of future system dynamics is not needed to determine present control action is expected to make this controller suitable for use in adaptive control schemes.

The approach of looking at the one-step-ahead minimization of the control action and of the state deviation from zero, as a formally equivalent estimation problem, has allowed to interpret the weighting matrices involved as error covariance matrices. This certainly facilitates the choice of these matrices.

Since the control gain depends on state uncertainty, complete separability does not hold. The implications of this feature have to be better evaluated.

The results obtained in the numerical tests are encouraging and here one should include those results obtained previously, in a heuristic basis, in application to nonlinear and time variant systems (Rios-Neto and Cruz, 1985; Ferreira, Rios-Neto and Venkataramen, 1985). However the numerical behavior of matrix $\bar{X}^P(i+1)$ (in Equation (11)) indicates that further investigation is needed to find ways of preventing the tendency it presents of getting nearly singular. In the cases tested, saturating the matrix from bellow was enough, but one should look at the results already available in state estimation to try to infer better approaches to treat this problem.

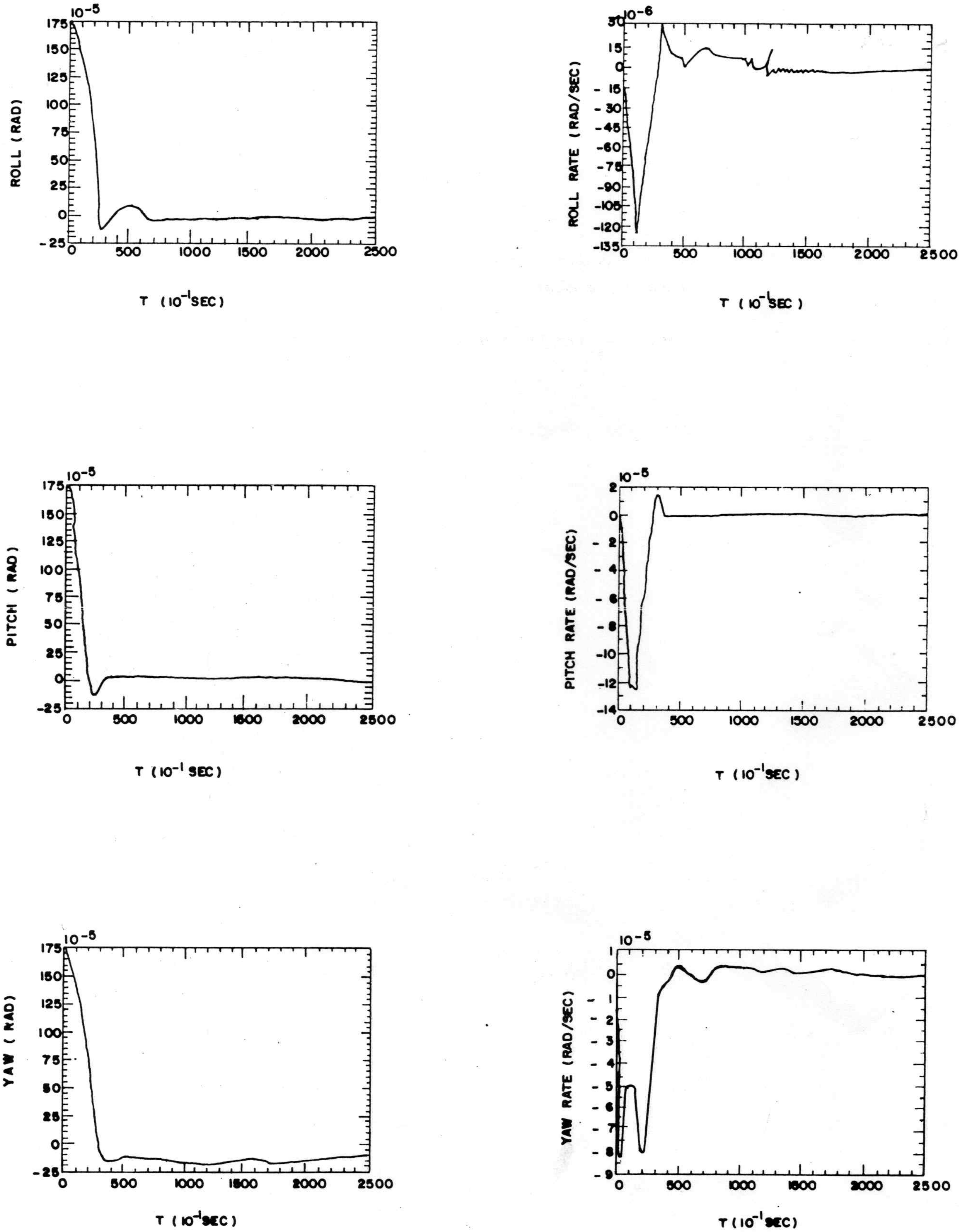

Fig. 1 – Simulation results for satellite threeaxis attitude control

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