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Effect of Force Model Errors on Short-Term Circular Orbit Estimations

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Abstract

The ephemerides of low-altitude artificial satellites generated by orbit propagation procedures are in general influenced by various types of disturbing force model errors. In the case of the gravitational force, model errors are caused by the incorrect truncation of gravity field as well as the errors in the included coefficients. In the case of atmospheric drag, they are caused by the uncertainty in the model parameters involved. Considering low-altitude circular orbits, this work aims at treating these two types of errors and developing a stochastic procedure to evaluate the order of magnitude of the accumulated global error in short-term orbit propagations. For the geopotential function, the linear estimation methods are combined with the spectral representation of terrestrial gravitation, and the statistical estimates of uncertainties in orbital elements are deduced in terms of their covariances. For the atmospheric drag force errors, the drag parameters are considered to be stochastic quantities and the statistical estimates of uncertainties in orbital elements are obtained the same way as in the geopotential case. The theories in both the cases are at first tested separately and then are combined and successfully applied to a satellite with the orbital and structural configuration similar to that of the proposed first Brazilian satellite.

Introduction

Low-altitude orbit propagation is one of the most important problems in the analysis and control of artificial satellite missions. The accuracy of the results of an orbit propagation process based on the special perturbation methods is very much affected by the errors in the perturbation model used in the orbital system dynamics and by the errors in the numerical procedure used for integrating the dynamic system. Consequently, in order to generate reliable and reasonably good data of the satellite ephemerides, it is necessary to evaluate the order of magnitude of the accumulated global error, owing to all these imperfections, in the processes of orbit propagations.

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One of the main errors in the perturbation model is due to the complicated form of the geopotential function. The terrestrial gravitational field, which is nonspherical, is in general represented by an infinite series expansion of spherical harmonics. For practical purposes, the series is truncated depending on the knowledge of the terms in series at that moment. The errors are then related to the incomplete knowledge of the coefficients included and to the ignored part. In general, the contribution due to the latter is smaller than that due to the former. However, in the orbit propagations where a low-degree model is used, the coefficients included are usually well-known so as to ignore the errors caused by their incorrectness, and the effect of errors due to omission, which are usually not known, become more significant [1]. Orbit estimation in the presence of gravitational anomalies has been the subject of various papers [2–7]. However, test cases spreading over a considerably wide range of altitudes and detailed analyses are lacking in most of these works. Besides this, some of these works are not aimed at attaining procedures which can be combined automatically with the software of guidance and control of dynamic systems where a stochastic approach is needed so as to couple the modeling errors easily with the state estimate errors [8–9]. Taking into account only the geopotential function errors, Gersten et al. [2] deduced the statistical estimates of the uncertainties in orbital elements in terms of their covariances considering the gravitational field as a statistical quantity [10] and using linear estimation methods. As one of the main aims, this paper, based upon the analytical expressions of Gersten et al. [2], applies the developed theory to some simple problems of satellites in a varied range of altitudes, compares the estimated errors with true errors and analyzes in detail the results.

The other significant error in orbit propagation procedures is due to the necessity of stochastic modeling of atmospheric properties. The air density used in calculating the drag force is a complicated function of altitude, longitude, latitude, solar and geomagnetic activities, and time. Moreover, the drag coefficient is not a well-determined parameter. In fact, these uncertainties complicate the trials to obtain a coupled solution with drag in the problem of motion of artificial satellites. All the same, there have been various papers treating the drag problem [11–13]. However, either these works have objectives different from that of this paper or the theories developed there are not appropriate for coupling with geopotential theory. Choosing some of the principal drag parameters whose uncertainty affects the accuracy in determining the drag force, another aim of this work is to treat these parameters as stochastic quantities so that the statistical estimates of uncertainties in orbital elements could be deduced in terms of their covariances and to apply the theory developed here, in combination with the geopotential theory, to a satellite with the orbital and structural configuration similar to that of the proposed first Brazilian satellite.

Stochastic processes to estimate the numerical integration errors have been developed elsewhere: in the case of single step methods [14], and also in the case of multistep methods [15]. Hence, this work is limited to treat only the errors in the perturbation model, i.e. the errors due to Earth's anomalous gravity and drag parameter uncertainty, in the case of short-term propagation of low altitude circular orbits. However, all the three types of the errors, including even long-term propagations in the case of numerical integration errors, were dealt with in the original work [16], which is a Ph.D. dissertation, on which this paper is based on, and an abridged version of this complete thesis work can be found elsewhere [17].

Earth's Anomalous Gravity

The stochastic differential equation which describes a non-linear dynamic system characterizing the orbit estimation problem may be written as [8]:

$$\dot{\mathbf{X}}(t) = \mathbf{f}(\mathbf{X}(t), t) + G(t)\Delta\mathbf{u}(t) \qquad t \geq t_0 \tag{1}$$

where $\mathbf{X}(t)$ is the orbit state vector, $G(t)$ is the control matrix and $\Delta\mathbf{u}(t)$ is a zero mean Gaussian process. The initial estimate $\mathbf{X}(t_0)$ is also supposed to have a Gaussian distribution with mean $\hat{\mathbf{X}}(t_0)$ and covariance matrix $P(t_0)$.

Let $\overline{\mathbf{X}}(t)$ be a reference trajectory with given $\overline{\mathbf{X}}(t_0)$, which satisfies the equation,

$$\dot{\overline{\mathbf{X}}}(t) = \mathbf{f}(\overline{\mathbf{X}}(t), t) \qquad t \geq t_0$$

Now defining $\Delta\mathbf{X}(t)$ as a deviation from the nominal trajectory, i.e.,

$$\Delta\mathbf{X}(t) \triangleq \mathbf{X}(t) - \overline{\mathbf{X}}(t)$$

and assuming that these deviations are small in the quadratic mean sense, one gets an approximate linear equation as:

$$\Delta\dot{\mathbf{X}}(t) = F(t, \overline{\mathbf{X}}(t_0))\Delta\mathbf{X}(t) + G(t)\Delta\mathbf{u}(t) \tag{2}$$

where

$$F(t, \overline{\mathbf{X}}(t_0)) = \left[\frac{\partial f_i(\overline{\mathbf{X}}(t), t)}{\partial X_j}\right]$$

which is a partial derivative matrix evaluated on the nominal trajectory.

The solution of the equation (2) is given by:

$$\Delta\mathbf{X}(t) = \phi(t, t_0)\Delta\mathbf{X}(t_0) + \int_{t_0}^{t} \phi(t, \tau)G(\tau)\Delta\mathbf{u}(\tau)\, d\tau \tag{3}$$

where $\phi(t, t_0)$ is the state transition matrix which satisfies the equation,

$$\dot{\phi}(t, t_0) = F(t, \mathbf{X}(t_0))\phi(t, t_0), \qquad \phi(t_0, t_0) = I$$

where I is an identity matrix.

Since $\Delta\mathbf{u}(t)$ is assumed to be a zero-mean Gaussian process, it can easily be seen from equation (3), by taking the expectance $\mathrm{E}\{\Delta\mathbf{X}(t)\}$, that $\Delta\mathbf{X}(t)$ is an unbiased process. Using this fact, the general expression of the covariance function of $\Delta\mathbf{X}(t)$ can be derived as:

$$\begin{aligned}\mathrm{COV}\{\Delta\mathbf{X}(t)\} &= \mathrm{E}\{\Delta\mathbf{X}\Delta\mathbf{X}^T\} \\ &= \phi(t, t_0)\mathrm{E}\{\Delta\mathbf{X}(t_0)\Delta\mathbf{X}^T(t_0)\}\phi^T(t, t_0) \\ &\quad + \int_{t_0}^{t}\int_{t_0}^{t} \phi(t, \tau)G(\tau)\mathrm{E}\{\Delta\mathbf{u}(\tau)\Delta\mathbf{u}^T(\eta)\}G^T(\eta)\phi^T(t, \eta)\, d\tau\, d\eta \end{aligned} \tag{4}$$

where τ and η are dummy variables. The cross terms have been cancelled here as the errors in the initial state are not correlated with errors in the perturbation model considered in subsequent propagation. In the equation (4), the first term on the right hand side of the equation is the familiar form of simple propagation of covariance matrix and the second term is an additive deweighting matrix which takes into account the umodeled part of the geopotential [18].

Orbital Elements and the Equations of Motion

For the description of the orbit (Fig. 1), the set $\{r, v, \beta, i, \Omega, \zeta\}$ of spherical elements [19], which is simple and useful even in very general cases [7] is chosen here. Here r is the radial distance between the satellite and the Earth's center, v is the satellite velocity, β is the angle between the satellite radius vector and the velocity vector, i is the orbital inclination, Ω is the longitude of ascending node and ζ is the angle between the maximum declination point and the instantaneous position of the satellite (measured in the positive direction of the satellite motion in orbital plane).

Considering the Earth as a uniform sphere without atmosphere, the equations of motion in the spherical elements chosen above, after substituting t by ζ as the independent variable so as to simplify the integration process, can be written as (see [2] or for detailed derivation [16]):

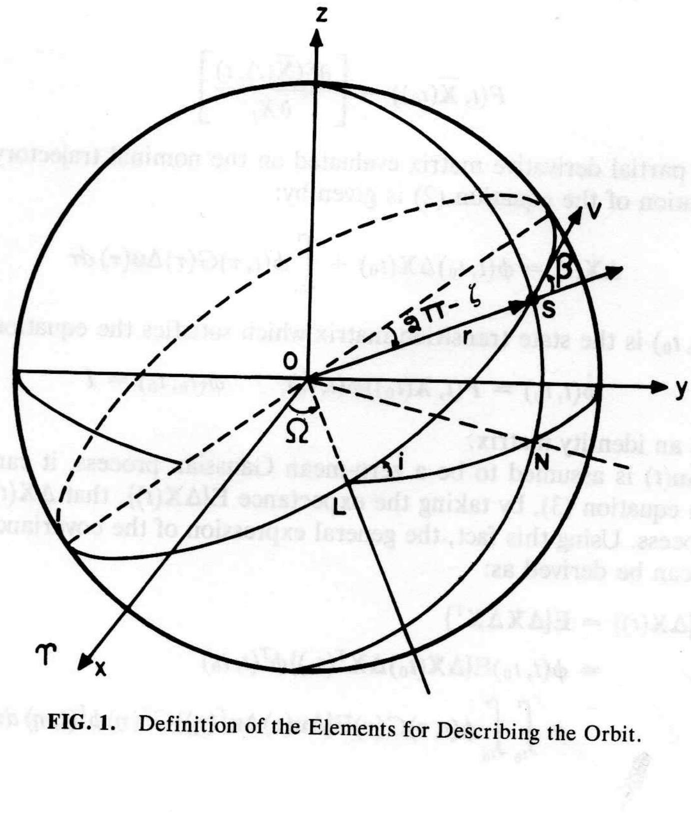

FIG. 1. Definition of the Elements for Describing the Orbit.

$$r' = r \cot \beta$$

$$v' = -[GM/(rv)] \cos \beta$$

$$\beta' = [GM/(rv^2)] - 1$$

$$i' = 0$$

$$\Omega' = 0$$

$$t' = (r/v) \sin \beta \qquad (5)$$

where prime indicates the derivative with respect to ζ.

It is known that in the absence of a disturbing function, all the orbits are Keplerian orbits. Introducing here the elements r_0, v_0, β_0, i_0, Ω_0 and t_0 as the Keplerian orbit elements, one defines:

$$r = r_0 + \Delta r$$

$$v = v_0 + \Delta v$$

$$\beta = \beta_0 + \Delta\beta$$

$$i = i_0 + \Delta i$$

$$\Omega = \Omega_0 + \Delta\Omega$$

$$t = t_0 + \Delta t \qquad (6)$$

where Δr, Δv, $\Delta\beta$, Δi, $\Delta\Omega$, Δt are the variations in the Keplerian orbital elements because of perturbations. Considering circular orbits, one can observe here that $\beta_0 = \pi/2$ and $r_o v_0^2 = GM$, where G is the universal gravitational constant and M is the mass of the Earth. The differential equations for these variations in the orbital elements are given by (see [2] or for detailed derivation [16]):

$$(\Delta r)' = -r_0 \Delta\beta$$

$$(\Delta v)' = v_0 \Delta\beta$$

$$(\Delta\beta)' = -\Delta r/r_0 - 2\Delta v/v_0$$

$$(\Delta i)' = 0$$

$$(\Delta\Omega)' = 0$$

$$(\Delta t)' = \Delta r/v_0 - r_0 \Delta v/v_0^2 \qquad (7)$$

Now one considers three mutually perpendicular components R, S and W of perturbing acceleration, R in the direction of the radius vector, S perpendicular to R in the orbital plane (positive in the direction of increasing longitude), and W perpendicular to the orbital plane. Based on the equations of Gauss, the inclusion of the disturbing function transforms the equations of motion (7) into the following set of equations (see [2] or for detailed derivation [16]:

$$
\begin{aligned}
(\Delta r)' &= -r_0\Delta\beta \\
(\Delta v)' &= v_0\Delta\beta + (r_0/v_0)S \\
(\Delta\beta)' &= -\Delta r/r_0 - 2\Delta v/v_0 - (r_0/v_0^2)R \\
(\Delta i)' &= -(r_0 \sin\zeta/v_0^2)W \\
(\Delta\Omega)' &= [r_0\cos\zeta/(v_0^2 \sin i)]W \\
(\Delta t)' &= \Delta r/v_0 - (r_0\Delta v)/v_0^2 - [r_0^2 \cos\zeta\cos i/(v_0^3 \sin i)]W
\end{aligned} \tag{8}
$$

Identification and Evaluation of the Terms in the Covariance Function

It can easily be seen that the equations (8) are equivalent to the matrix equation (2) except that the independent variable in the former is ζ instead of t. Comparing these two equations, one obtains:

$$
\Delta\mathbf{X} = \begin{bmatrix} \Delta r \\ \Delta v \\ \Delta\beta \\ \Delta i \\ \Delta\Omega \\ \Delta t \end{bmatrix} \qquad \Delta\mathbf{u} = \begin{bmatrix} R \\ S \\ W \end{bmatrix} \tag{9a}
$$

$$
F = \begin{bmatrix} 0 & 0 & -r_0 & 0 & 0 & 0 \\ 0 & 0 & v_0 & 0 & 0 & 0 \\ -1/r_0 & -2/v_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1/v_0 & -r_0/v_0^2 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{9b}
$$

Taking ζ as the independent variable in the place of t, the covariance matrix of uncertainties in the orbital elements, given by the equation (4), may be written as:

$$
\begin{aligned}
\mathrm{COV}\{\Delta\mathbf{X}(\zeta)\} &= \phi(\zeta,\zeta_0)P(\zeta_0)\phi^T(\zeta,\zeta_0) \\
&\quad + \int_{\zeta_0}^{\zeta}\int_{\zeta_0}^{\zeta} \phi(\zeta,\tau)G(\tau)\mathrm{E}\{\Delta\mathbf{u}(\tau)\Delta\mathbf{u}^T(\eta)\}G^T(\eta)\phi^T(\zeta,\eta)\,d\tau\,d\eta
\end{aligned} \tag{10}
$$

The transition matrix $\phi(\zeta,\tau)$ within the double integral of the equation (10) can be obtained directly from the transition matrix equation, with matrix F given as in the expression (9b). The matrices $\phi(\zeta,\tau)$ and $G(\tau)$ of the equation (10) can be expressed as:

$$
\phi(\zeta,\tau) = \begin{bmatrix} 2-\mathrm{c}(z) & 2C_1[1-\mathrm{c}(z)] & -r_0\mathrm{s}(z) & 0 & 0 & 0 \\ C_2[\mathrm{c}(z)-1] & 2\mathrm{c}(z)-1 & v_0\mathrm{s}(z) & 0 & 0 & 0 \\ -C_3\mathrm{s}(z) & -C_4\mathrm{s}(z) & \mathrm{c}(z) & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ C_5[3z-2\mathrm{s}(z)] & C_6[3z-4\mathrm{s}(z)] & -2C_1[1-\mathrm{c}(z)] & 0 & 0 & 1 \end{bmatrix} \tag{11a}
$$

$$G^T(\tau) = \begin{bmatrix} 0 & 0 & -C_6 & 0 & 0 & 0 \\ 0 & C_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -C_6 s(\tau) & C_6 c(\tau)/s(i) & -C_1 C_6 c(\tau) c(i)/s(i) \end{bmatrix} \quad (11b)$$

where $z = \zeta - \tau$; $c(\cdot) = \cos(\cdot)$; $s(\cdot) = \sin(\cdot)$; $C_1 = r_0/v_0$; $C_2 = v_0/r_0$; $C_3 = 1/r_0$; $C_4 = 2/v_0$; $C_5 = 1/v_0$; $C_6 = C_1/v_0$.

Gravity Error Covariance

Let $\mathbf{\Delta u}(t)$ be a vector of random errors in gravity, with Gaussian distribution, given by:

$$\mathbf{\Delta u}(t) = \begin{bmatrix} \Delta u_R(t) \\ \Delta u_S(t) \\ \Delta u_W(t) \end{bmatrix} \quad (12)$$

where subscripts R, S and W denote the radial, transversal and normal directions, respectively. The orthogonal properties of spherical harmonics assure that the radial, transversal and normal components of the errors in gravity are not correlated among themselves [7]. Now, one defines ψ as an angular distance (central angle) of an arbitrary point $P'(\theta', \lambda')$ with respect to a fixed point $P(\theta, \lambda)$ on the surface of a sphere of radius r_0 as shown in Fig. 2, where θ and θ' are the polar distances, and λ and λ' are the longitudes of P and P', respectively. Extending the results of Kaula [10] to the altitudes of satellites, the diagonal function of auto-covariances of the errors in modeling gravity on a sphere of arbitrary radius r is

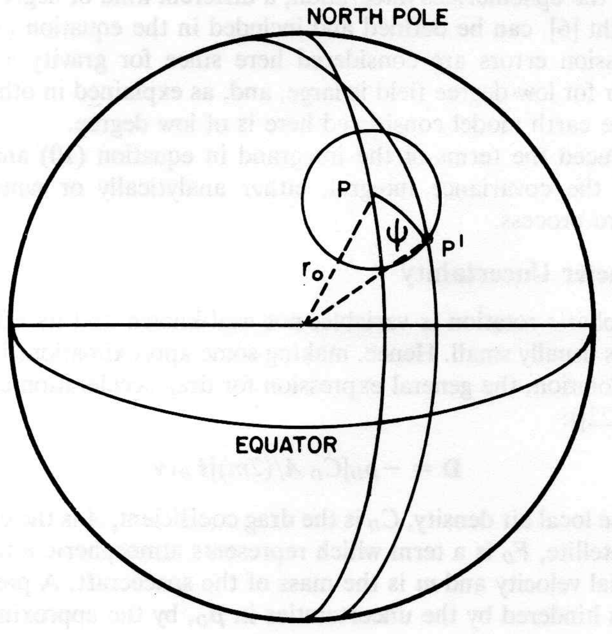

FIG. 2. Definition of Angle ψ between P and P' on a Spherical Surface.

given by ([2] and [16]):

$$C(\psi) = \mathrm{E}\{\Delta\mathbf{u}(\tau)\Delta\mathbf{u}^T(\eta)\} = \begin{bmatrix} C_R(\psi) & 0 & 0 \\ 0 & C_S(\psi) & 0 \\ 0 & 0 & C_W(\psi) \end{bmatrix} \tag{13}$$

where

$$\begin{aligned} C_R(\psi) &= \sum_{n=2}^{\infty} \left(\frac{n+1}{n-1}\right)^2 \left(\frac{R_T}{r}\right)^{2n+4} \sigma_n^2 P_n(\cos\psi(\tau-\eta)) \\ C_S(\psi) &= \sum_{n=2}^{\infty} \frac{n(n+1)}{2(n-1)^2} \left(\frac{R_T}{r}\right)^{2n+4} \sigma_n^2 \left\{ P_n(\cos\psi(\tau-\eta)) - \frac{P_n^2(\cos\psi(\tau-\eta))}{(n+1)n} \right\} \\ C_W(\psi) &= \sum_{n=2}^{\infty} \frac{n(n+1)}{2(n-1)^2} \left(\frac{R_T}{r}\right)^{2n+4} \sigma_n^2 \left\{ P_{n-1}(\cos\psi(\tau-\eta)) + \frac{P_{n-1}^2(\cos\psi(\tau-\eta))}{n(n+1)} \right\} \end{aligned} \tag{14}$$

Here R_T is the Earth's equatorial radius, P_n and P_n^2 are, respectively, Legendre's polynomials and associated functions. The quantity σ_n^2 is called the degree variance which is defined as [20]:

$$\sigma_n^2 = \sum_{m=0}^{n} (\bar{a}_{nm}^2 + \bar{b}_{nm}^2) \tag{15}$$

where $\bar{a}_{nm}$ and $\bar{b}_{nm}$ are the geopotential coefficients which are ignored in the integration of ephemerides. In order to treat the errors in the geopotential coefficients used in the ephemerides integration, a different kind of degree variance, as given by Wright [6], can be defined and included in the equation (14). However, only the omission errors are considered here since for gravity anomalies, the omission error for low-degree field is large, and, as explained in other sections of this paper, the earth model considered here is of low degree.

Having deduced the terms of the integrand in equation (10) analytically, the evaluation of the covariance integral, either analytically or numerically, is a straightforward process.

Drag Parameter Uncertainty

The atmospheric rotation is variable, not well-known and its effect on atmospheric drag is usually small. Hence, making some approximations in treating the atmospheric rotation, the general expression for drag acceleration can be written as ([21] and [22]):

$$\mathbf{D} = -\rho_D[C_D A/(2m)]F_D v\mathbf{v} \tag{16}$$

where ρ_D is the local air density, C_D is the drag coefficient, A is the cross-sectional area of the satellite, F_D is a term which represents atmospheric rotation, $\mathbf{v}$ is the satellite inertial velocity and m is the mass of the spacecraft. A precise determination of $\mathbf{D}$ is hindered by the uncertainties in ρ_D, by the approximations in the ballistic coefficient B $(= C_D A/(2m))$ and by the time variance of the factor F_D.

Writing the differential equation of the dynamic system as:

$$\dot{\mathbf{X}}(t) = \mathbf{f}(\mathbf{X},\mathbf{u}_1,t) + \mathbf{g}(\mathbf{X},\mathbf{u}_2,t)$$

where the function **f** contains the gravitational potential terms, $\mathbf{u}_1$, and the function **g** contains the drag terms $\mathbf{u}_2$, and defining two zero mean Gaussian processes,

$$\Delta\mathbf{u}_1 = \mathbf{u}_1 - \bar{\mathbf{u}}_1$$

$$\Delta\mathbf{u}_2 = \mathbf{u}_2 - \bar{\mathbf{u}}_2$$

where the values with overbars are nominal values, one will have:

$$\Delta\dot{\mathbf{X}} = \left(\frac{\partial\mathbf{f}}{\partial\mathbf{X}} + \frac{\partial\mathbf{g}}{\partial\mathbf{X}}\right)\Delta\mathbf{X} + \frac{\partial\mathbf{f}}{\partial\mathbf{u}_1}\Delta\mathbf{u}_1 + \frac{\partial\mathbf{g}}{\partial\mathbf{u}_2}\Delta\mathbf{u}_2 \tag{17}$$

The solution of the equation (17) is given by:

$$\Delta\mathbf{X}(t) = \phi(t,t_0)\Delta\mathbf{X}(t_0) + \int_{t_0}^{t}\phi(t,\tau)G(\tau)\Delta\mathbf{u}_1(\tau)\,d\tau + \int_{t_0}^{t}\phi(t,\tau)H(\tau)\Delta\mathbf{u}_2(\tau)\,d\tau \tag{18}$$

where

$$G \triangleq \frac{\partial\mathbf{f}}{\partial\mathbf{u}_1}\Bigg|_{\substack{\mathbf{X}=\overline{\mathbf{X}}\\ \mathbf{u}_1=\bar{\mathbf{u}}_1}} \qquad H \triangleq \frac{\partial\mathbf{g}}{\partial\mathbf{u}_2}\Bigg|_{\substack{\mathbf{X}=\overline{\mathbf{X}}\\ \mathbf{u}_2=\bar{\mathbf{u}}_2}}$$

and $\phi(t,t_0)$ is the state transition matrix which satisfies the equation,

$$\dot{\phi}(t,t_0) = F(t,\mathbf{X}(t_0))\phi(t,t_0), \qquad \phi(t_0,t_0) = I$$

with

$$F(t,\mathbf{X}(t_0)) = \frac{\partial\mathbf{f}}{\partial\mathbf{X}} + \frac{\partial\mathbf{g}}{\partial\mathbf{X}}$$

The covariance matrix of uncertainties in the orbital elements, $\Delta\mathbf{X}(t)$, is then given by:

$$\begin{aligned}\mathrm{COV}\,\{\Delta\mathbf{X}(t)\} &= \phi(t,t_0)\mathrm{E}\{\Delta\mathbf{X}(t_0)\Delta\mathbf{X}^T(t_0)\}\phi^T(t,t_0)\\ &+ \int_{t_0}^{t}\int_{t_0}^{t}\phi(t,\tau)G(\tau)\mathrm{E}\{\Delta\mathbf{u}_1(\tau)\Delta\mathbf{u}_1^T(\eta)\}G^T(\eta)\phi^T(t,\eta)\,d\tau\,d\eta\\ &+ \int_{t_0}^{t}\int_{t_0}^{t}\phi(t,\tau)H(\tau)\mathrm{E}\{\Delta\mathbf{u}_2(\tau)\Delta\mathbf{u}_2^T(\eta)\}H^T(\eta)\phi^T(t,\eta)\,d\tau\,d\eta\end{aligned} \tag{19}$$

ignoring the cross-terms on the basis of the hypothesis that the initial state errors, the errors in geopotential model and the errors in drag parameters are not correlated among themselves.

The second and third terms on the right hand side of the expression (19) are called additive deweighting matrices which take into account the part of the geopotential function which was not modeled (same as the term of the equation (4) but for a slight difference in the transition matrix) and the uncertainties in drag parameters, respectively. The detailed deduction of equation (19) and the resolution of its terms are clearly explained elsewhere [16]. Concisely, writing the dynamic equations of motion in terms of the same spherical set used in the case of gravitational anomaly, the system matrix F in this case of combining atmospheric drag with geopotential, can be seen to be [16]:

$$F = \begin{bmatrix} 0 & 0 & -r_0 & 0 & 0 & 0 \\ -Cv_0 c(i_0) & -Cr_0 c(i_0) & v_0 & Cr_0 v_0 s(i_0) & 0 & 0 \\ -C_3 & -C_4 & -Cr_0(1 - c(i_0)) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ C_5 & -C_6 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{20}$$

where $C = \rho_D C_D (A/m) F_D$, and c(·), s(·) and C_i $(i = 3, 4, 5, 6)$ are as defined in the equations (11).

The problem of finding the corresponding transition matrix can then be solved analytically by using the methods of homogeneous and non-homogeneous linear systems of equations.

Now, in developing the covariance matrix of errors in drag parameters, $E\{\Delta\mathbf{u}_2(\tau)\Delta\mathbf{u}_2^T(\eta)\}$, defining,

$$\Delta\mathbf{u}_2 = \begin{bmatrix} \rho_D - \bar{\rho}_D \\ B - \bar{B} \\ F_D - \bar{F}_D \end{bmatrix} \tag{21}$$

the uncertainties in ρ_D, B and F_D are modelled by zero-mean random variables with Gaussian distributions whose standard deviations are such that

$$3\sigma_i = \varepsilon_i$$

where ε_i $(i = 1, 2, 3)$, are the orders of magnitude of these errors.

Thus, the uncertainties in these parameters are modelled by random variables with statistical properties defined as:

$$E\{\varepsilon_\rho\} = 0; \qquad E\{\varepsilon_\rho^2\} = \sigma_1^2 = \varepsilon_1^2/9$$
$$E\{\varepsilon_B\} = 0; \qquad E\{\varepsilon_B^2\} = \sigma_2^2 = \varepsilon_2^2/9$$
$$E\{\varepsilon_F\} = 0; \qquad E\{\varepsilon_F^2\} = \sigma_3^2 = \varepsilon_3^2/9$$

One will then have:

$$E\{\Delta\mathbf{u}_2\Delta\mathbf{u}_2^T\} = E\left\{\begin{matrix} \varepsilon_\rho^2 & 0 & 0 \\ 0 & \varepsilon_B^2 & 0 \\ 0 & 0 & \varepsilon_F^2 \end{matrix}\right\}$$

In the case of ρ_D, the order of magnitude of its error, ε_1, is obtained by comparing the observed solar activity indices with the indices predicted by NASA Marshall Space Flight Center [23]. Considering these values for the period between May 1981 and September 1986, which covers maximum, medium and minimum solar activity, corresponding exospheric temperature values and density values are calculated using the density model CIRA-1972 [24], and the comparison estimates the order of magnitude of the error in density to be 10^{-13} kg/m^3.

In the case of F_D, the expression is given by [22]

$$F_D = \left[1 - \frac{r_p}{v_p} \omega_T \cos i_0 \right]^2$$

where r_p and v_p represent the values of position and velocity at perigee and ω_T represents the angular velocity of the atmosphere. Here, the uncertainty in F_D is mainly due to ω_T. It can reasonably be assumed that the mean rotation rate of the atmosphere, being equal to Earth's rotation rate at the altitude of 200 km, increases by 30% at the altitude of 250 km and then decreases to 80% the Earth's rotation rate at the altitude of 600 km [25]. With this, the order of magnitude of the uncertainty in the parameter F_D may be estimated to be 0.01.

The order of magnitude of the error in the ballistic coefficient is calculated based on the given configuration of the satellite in the next section.

Finally, the double integrals appearing in the covariance matrix expression (19) are evaluated numerically, whose details are given in the next section.

Results and Discussion

In the case of estimating errors in modeling the gravitational function, three test problems have been chosen. The orbits considered are circular with an inclination of 25° and altitudes of 400, 600 and 800 km. The basic idea behind the selection of these test problems is to have results applicable to planned future Brazilian satellites which will have low circular orbits.

As the final aim is to combine the geopotential theory with that of drag, and as the double integral corresponding to drag in the expression (19) has to be solved numerically, the double integral corresponding to gravitational function also has been evaluated here numerically. For this evaluation, tests have been made with various quadratures such as QUANC8, an automatic adaptive routine based on the 8-panel Newton-cotes rule [26], Gauss-Patterson quadrature [27], and, finally a composite formula of Simpson [28] has been chosen after analyzing the processing time and the numerical errors involved in all the quadratures considered. At the initial point of orbit propagation, it is assumed that there are no orbit errors; and the integration is done in one orbit.

For comparison purposes, a true error has been generated by using a numerical generator [29] which integrates the perturbed equations of motion using a predictor-corrector of order which goes up to 12. Here, a geopotential model complete to degree and order 30 was considered to be the validation model and a model complete to degree and order 6 was considered to be the working model. For computing the corresponding estimated error, to maintain the coherence, the summations in equation (14) have been taken from 7 to 30. The comparisons of estimated errors with true errors in all the three test cases are shown in Figs. 3 through 5. The dashed curves in all the figures represent the $\pm\sigma$ (standard deviation) variation of the estimated error, and the continuous curve represents the true error. The true error curve has a discontinuous look as the errors have been

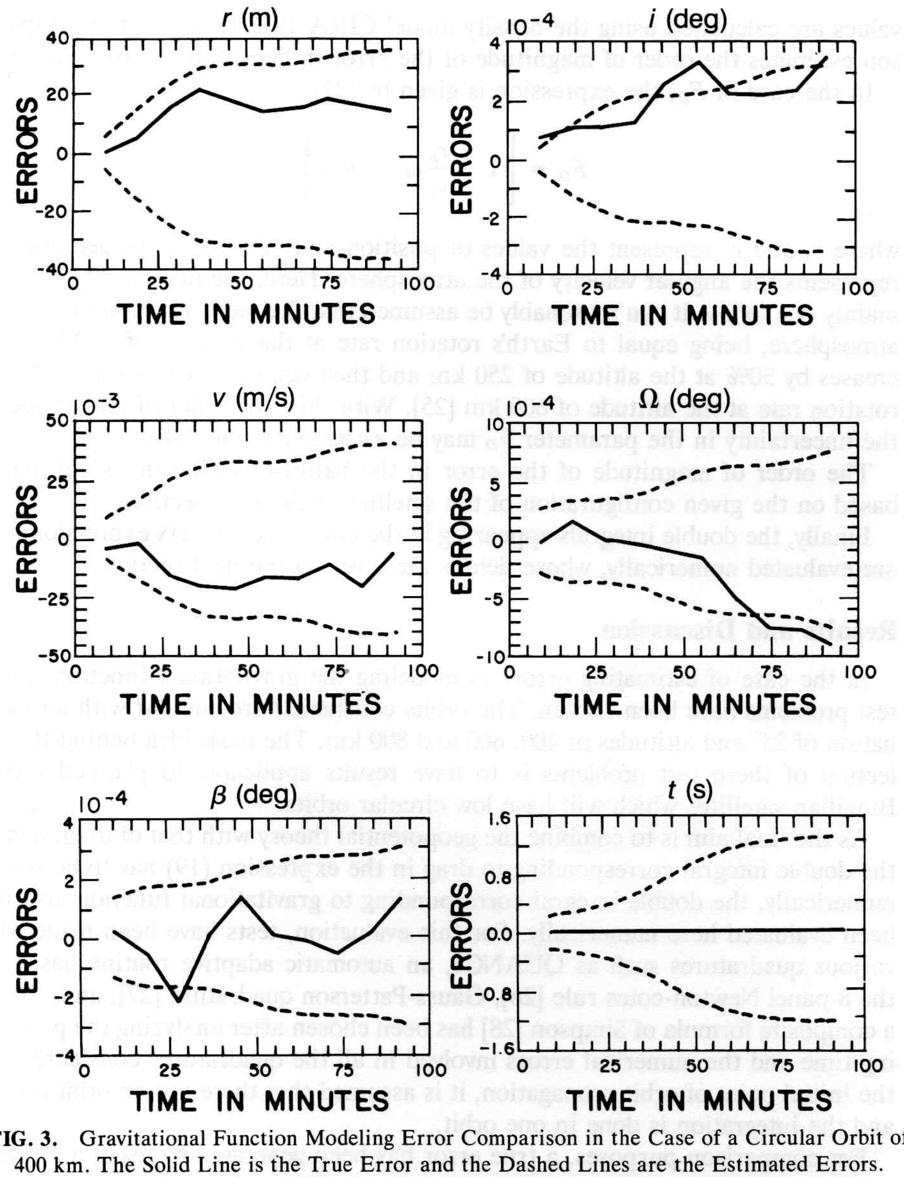

FIG. 3. Gravitational Function Modeling Error Comparison in the Case of a Circular Orbit of 400 km. The Solid Line is the True Error and the Dashed Lines are the Estimated Errors.

calculated at discrete points. Analyzing the Figs. 3 to 5, it can be seen that in all the cases, the error estimates are good and conservative.

Now in the case of drag parameter uncertainty, a hypothetical satellite in a circular orbit of 700 km with geometrical structure similar to the first Brazilian satellite was chosen. As the key note of this work is only to have a preliminary evaluation of the theory developed, the inclination of the orbit has been chosen to be 1° so that the system matrix F given in equation (20) could be approximated to

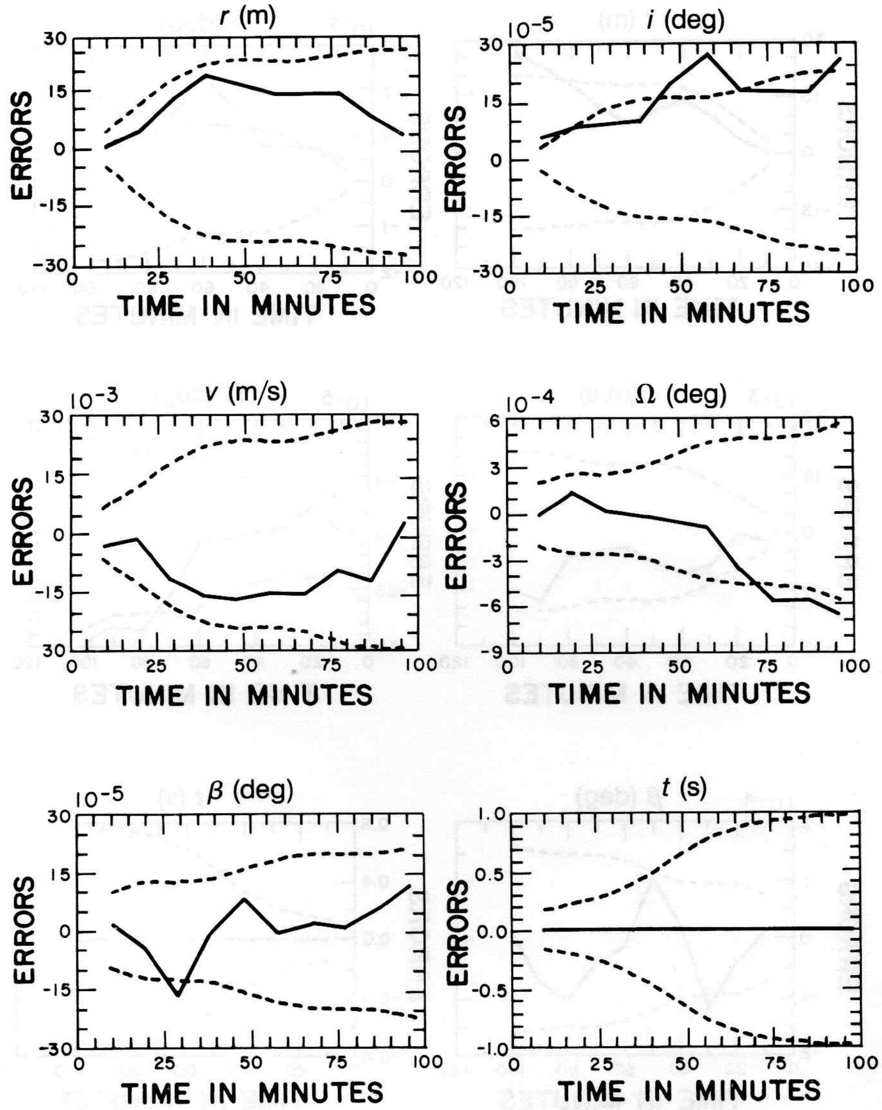

FIG. 4. Gravitational Function Modeling Error Comparison in the Case of a Circular Orbit of 600 km. The Solid Line is the True Error and the Dashed Lines are the Estimated Errors.

the system matrix in the case of geopotential alone, for which the transition matrix is available analytically.

For the given satellite, after calculating the most realistic drag coefficient [30] and after consulting other data of the satellite, the uncertainty in the parameter B is found to be the order of 10^{-3} m^2/kg [16]. The uncertainty in the other two parameters was already computed as explained in the previous section.

Having computed all the necessary parameters, to figure out the true error due to drag parameter uncertainty, the orbit of the satellite has been integrated in one

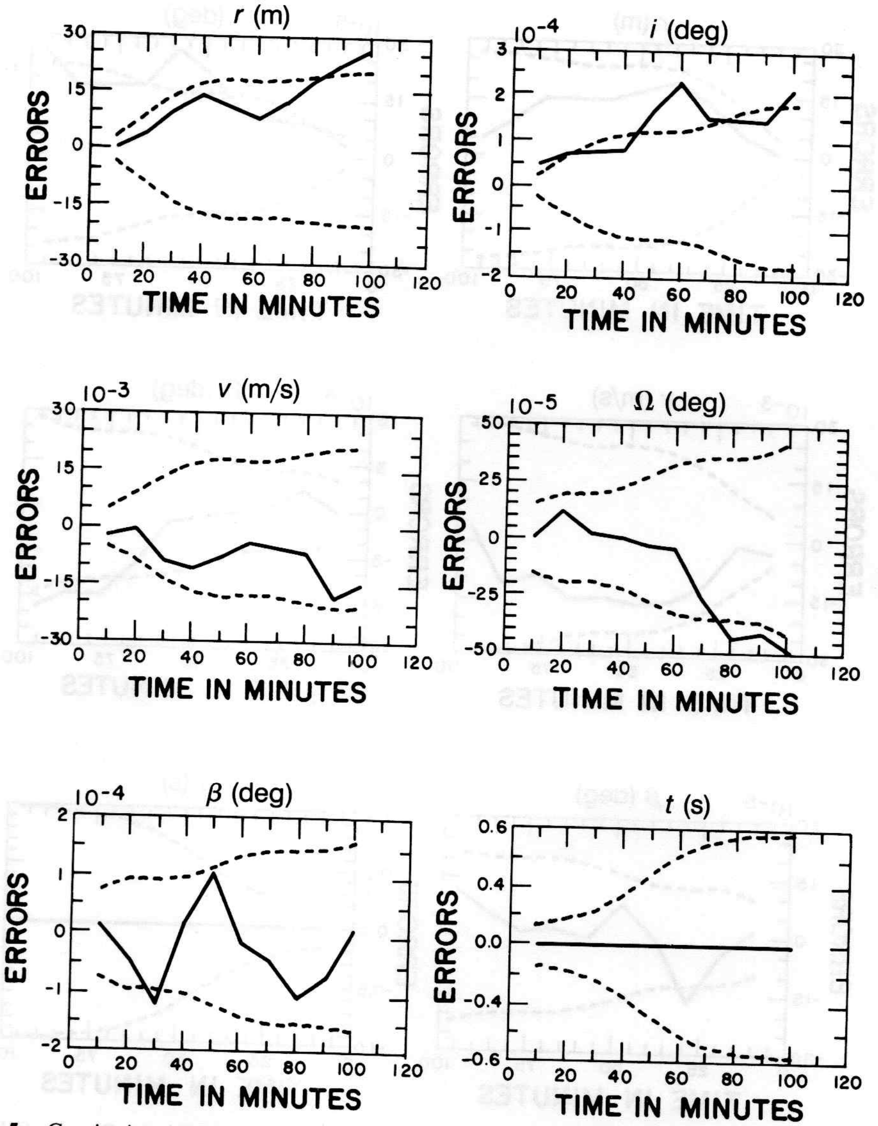

FIG. 5. Gravitational Function Modeling Error Comparison in the Case of a Circular Orbit of 800 km. The Solid Line is the True Error and the Dashed Lines are the Estimated Errors.

revolution, once with the nominal values of ρ_D, B and F_D and then with the most realistic values of these parameters, and the difference has been taken as the true error. In both the cases, the geopotential model used was the same.

For computing the corresponding estimated error, the double integral corresponding to the drag parameter uncertainty in equation (19) is evaluated numerically by applying the composite formula of Simpson used in the case of the geopotential.

The comparison of the estimated errors with true errors is shown in Fig. 6. The results in the case of the elements i and Ω are not shown because the error

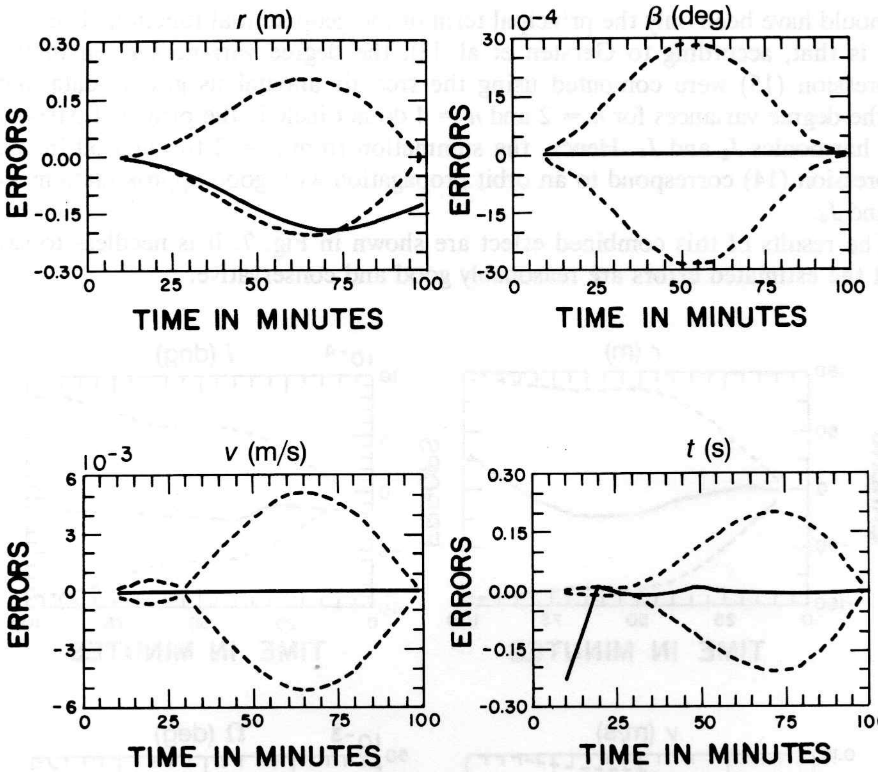

FIG. 6. Drag Force Parameter Error Comparison in the Case of a Circular Orbit of 700 km. The Solid Line is the True Error and the Dashed Lines are the Estimated Errors.

estimation theory was based upon a hypothesis that the effect of drag on these elements is negligible. In all other elements for which the results are shown, the estimated errors are very close to the true errors. Except in the case of *r*, the estimated errors are a little more conservative in the middle of the orbit but at the end of the orbit they present a better behavior.

Now, to obtain the combined effect of the errors caused by Earth's gravitational anomaly and drag parameter uncertainty, at first, a geopotential model complete to degree and order 4, and a drag model with mean values of drag parameters have been taken to constitute a working perturbation model, and the integration has been performed in one revolution. Then, a geopotential model complete to degree and order 30, and a drag model with most realistic values of the parameters chosen are supposed to form the validation or actual perturbation model, and the integration has been performed the same way as in the case of working perturbation model. The difference between these two results gave the true error. Then the error estimate has been computed by evaluating the two double integrals of equation (19), with summations in equation (14) going from 2 to 30 and using the nominal mean values of drag parameters.

Here one should make a note of a subtlety involved in using a geopotential model truncated to degree and order 4 for computing the true error, whereas it should have been only the principal term of the geopotential function. The reason is that, according to Gersten et al. [2], the degree variances given in the expression (15) were computed using the free-air anomalous gravity data and so the degree variances for $n = 2$ and $n = 4$ do not include the principal parts of the harmonics J_2 and J_4. Hence, the summation from $n = 2$ through 30 in the expression (14) correspond to an orbit propagation with good approximations of J_2 and J_4.

The results of this combined effect are shown in Fig. 7. It is needless to say that the estimated errors are reasonably good and conservative.

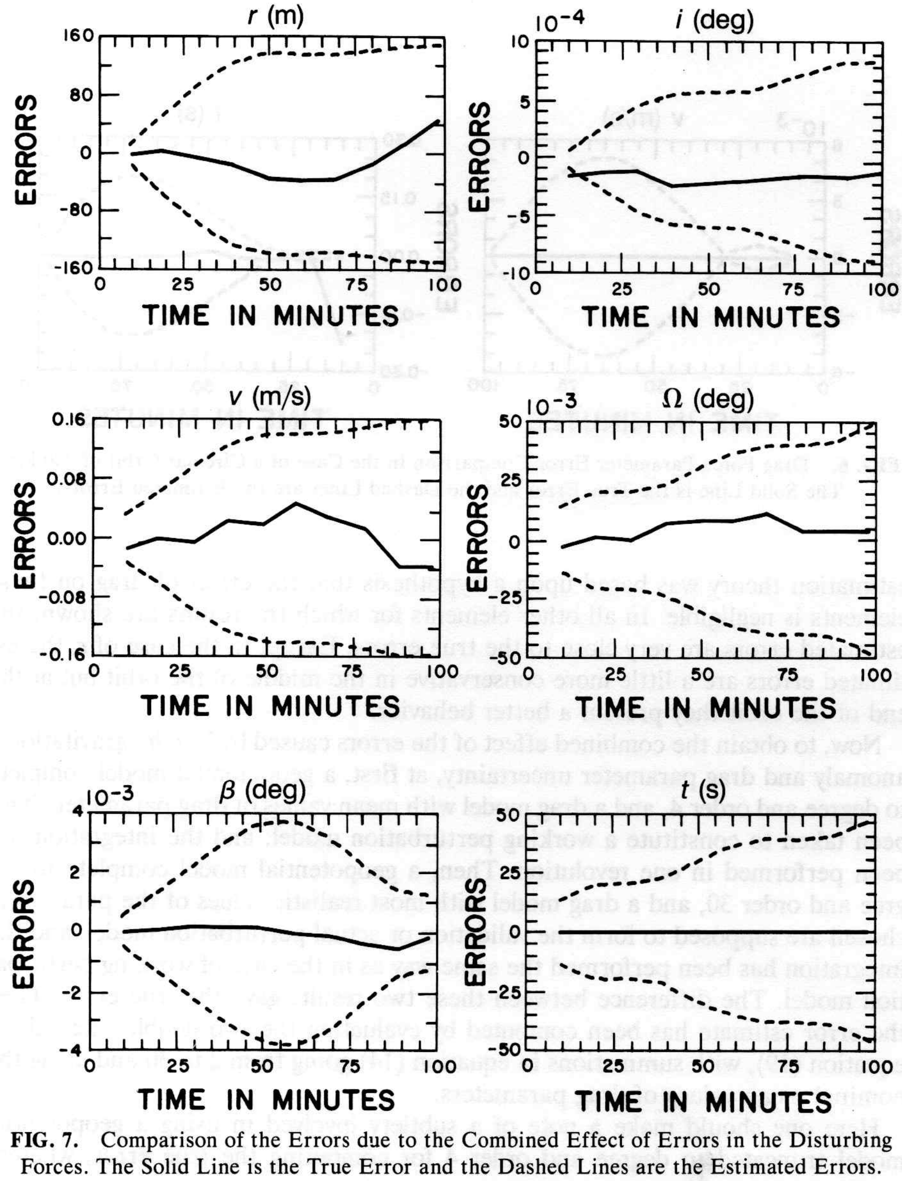

FIG. 7. Comparison of the Errors due to the Combined Effect of Errors in the Disturbing Forces. The Solid Line is the True Error and the Dashed Lines are the Estimated Errors.

Conclusions and Proposals for Future Extensions

In general terms, one can conclusively note that the theory used to evaluate the influence of the errors in modeling gravitational function and in drag force works well and gives satisfactory results in the short-term propagations of low-altitude circular orbits of artificial satellites. It can also be said that the spherical element set used in this work is very much appropriate for this type of study.

Specifically, in the case of geopotential function modeling errors, the theory is validated, as a preliminary evaluation, in various short-term circular orbit cases. It should be noted that only short term (one period) propagations of the orbit were considered not because of the shortcomings or limitations of the theory but because of slow and not very sophisticated quadratures used in this work. An analytical evaluation of the double integral would have made it possible to consider a long-term propagation [2], but with the aim of combining the geopotential theory with the drag theory, the evaluation has been done here numerically. In the case of drag force errors, one can assert that the theory developed here is suitable to be combined with the geopotential function theory.

In the case of modeling the errors due the Earth's anomalous gravity, though the basic expressions already exist [2, 10], this work showed an application of the theory in various examples and provided an analysis by comparing the estimated errors with the true errors, which has not been done before. Besides this, in the case of modeling the errors due to the drag parameter uncertainty, in this work, the theory has been developed, an application has been shown, the estimates are compared with the true errors and the coupling with the geopotential theory has been done. For complete derivation of the relating expressions, Kondapalli [16] may be looked into.

As a future development, at first, a faster quadrature should be found so as to validate the theory in long-term propagations in both geopotential and atmospheric drag cases separately and also when coupled. The geopotential theory may be tested in the case of some near-circular orbits following the guidelines suggested by Wright [6] and necessary changes for similar orbits should be done in atmospheric drag theory too. Some problems, where errors due also to the incorrectness of coefficients included in the geopotential model are treated, may be tested. The authors believe that this will need only a little more effort, hopefully. However, one will have to grapple a little with the problem of extending the theory to elliptical problems in the guidelines suggested by Giacaglia [7]. In drag theory, a comparison should be made between estimating the ballistic coefficient as a part of the state vector and calculating the drag model error taking into account the uncertainty in the ballistic coefficient, in order to study the relative error reduction in these cases. The case of attaining a general drag theory, though not an impossible task, needs some long term research. To achieve the objective, the methodology applied in geopotential theory should be tested in the case of drag theory also for further extensions.

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