LINEAR PROGRAMMING AND SUBOPTIMAL SOLUTIONS OF DYNAMICAL SYSTEMS CONTROL PROBLEMS

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ABSTRACT

Linear Programing is used to obtain a direct search procedure for parameter optimization in the numerical solution of dynamical systems control problems including those where state inequality constraints are present. The procedure applies to optimal control problems where the Raileigh-Ritz method is used to obtain a suboptimal approximation, imposing a particular mathematical form to the control function, making it dependent upon a finite number of parameters. Due to its characteristic of reducing the problem, in each iteration, to one of optimization of a direct search increment vector of parameters, inherent plant parameters are easily treated; exploring the fact that this characteristic implies having perturbations in the final states which explicity depend upon the control parameters, it is shown that state inequality constraints can be included without the need of using penalty function methods.

Keywords: Suboptimal Control, Raileigh-Ritz Method, Trajectory Optimization, Linear Programing.

1. INTRODUCTION

Recently, the approach of adopting a suboptimal approximation for the numerical solution of dynamical systems optimal control problems employing the Raileigh-Ritz method has been used, by imposing a particular mathematical form to the control, making it dependent upon a certain number of parameters (Refs. 1-6). In this paper, a first order direct search procedure is used in the solution of the restricted problem generated by the suboptimal approximation of the control. The problem associated to a typical iteration is constructed such as to allow a solution by linear programing (Refs. 3-5) and an extension of results presented in Ref. 5 is made to include the treatment of state inequality constraints, leading to a procedure that has the characteristic of being simple to formulate and to implement on the computer.

To test the behavior of the procedure, specially concerning the aspect of the suboptimal control function form, a simplified Earth-Mars minimum time orbit transfer, with low thrust of fixed magnitude and controlled direction, is considered. To simulate a situation where state inequality constraints are present, an upper bound limit is imposed to radial velocity in some of the numerical tests.

2. SUBOPTIMAL PROBLEM

The optimal control problem to be treated is to find the control function u(t), in the interval $[t_0, t_f]$, such as to minimize the index of performance and to comply with the following constraints:

$$IP = IP(x_f, t_f) \tag{1}$$

$$\dot{x} = f(x,u,t) \tag{2}$$

$$M(x_f, t_f) = 0 \tag{3}$$

where x is the nx1 state vector; $x(t_0)$ and t_0 are given; x_f is the final state corresponding to final t_f; u is the qx1 control vector; and M(.) is the mx1 vector constraint function of final conditions on state and time; IP is the index of performance.

Whenever state inequality constraints are present, in correspondence to each constraint $S_i(x,t) \leq 0$, additional state variables are defined to have:

$$\dot{x}_{n+i} = S_i(x,t), \quad \text{if} \quad S_i(x,t) \geq 0 \tag{4}$$

$$\dot{x}_{n+i} = 0, \quad \text{if} \quad S_i(x,t) < 0 \tag{5}$$

$$M_{m+i}(x_{n+i}(t_f)) = x_{n+i}(t_f) = 0 \tag{6}$$

Supposing that, whenever necessary, the previous extension was made, and if u(t) is substituted by u(a,t) or, in a general form, by u(a,x,t), the problem becomes:

$$IP = IP(x_f, a) \tag{7}$$

$$\dot{x} = f(x,a,t) \tag{8}$$

$$M(x_f, a) = 0 \tag{9}$$

where $x(t_0)$ and t_0 are given or defined as function of the parameters to be optimized; a is the gx1 vector of the parameters to be optimized, which may include inherent plant parameters; and t_f was substituted by the parameter a_g.

3. TYPICAL ITERATION ASSOCIATED PROBLEM

From a linear perturbation in the constraint equations and in the index of performance of the problem of Eqs. 7-9, it results that:

$$\Delta M = M_{x_f} x_{f_a} \Delta a + M_a \Delta a \quad (10)$$

$$\Delta IP = IP_{x_f} x_{f_a} \Delta a + IP_a \Delta a \quad (11)$$

where the subscripts indicate partial derivatives of M, x_f and IP with respect to x_f and a. To satisfy the criterion of getting closer to the suboptimum solution with sufficiently small increments, it is taken:

$$\Delta M = \alpha M \, , \qquad -1 \leq \alpha < 0 \quad (12)$$

$$\Delta IP \geq \gamma(/IP/+1) \, , \quad \gamma < 0 \quad (13)$$

where the condition given by Eq. 13, aside from contributing for small increments, translates that it is not always possible to get closer to constraint satisfaction and yet to decrease the index of performance.

To choose the problem associated to a typical iteration, which will lead to a scheme for the determination of the search increment, two aspects have to be considered. First, in the limits given by Eq. 13, ΔIP should be minimized. Second, to increase convergence speed it is necessary to move along a direction which is close to constraint gradient direction, that is, a norm of the increment vector Δa should be minimized. Based on these considerations, and from Eqs. 10-13, the associated optimization problem is taken as the minimization of

$$G = \sum_{i=1}^{g} \omega_i /\Delta a_i/ + \omega \Delta IP, \quad \omega > 0, \; \omega_i > 0 \quad (14)$$

subject to

$$M_{x_f} x_{f_a} \Delta a + M_a \Delta a = \alpha M \quad (15)$$

$$IP_{x_f} x_{f_a} \Delta a + IP_a \Delta a \geq \gamma(/IP/+1) \quad (16)$$

To formulate the problem in the usual form of linear programing, the following change of variables is made:

$$\Delta a_i = s_i - s_{g+i} \, , \; s_i \geq 0, \; s_{g+i} \geq 0, \; i=1,2,..,g \quad (17)$$

where $s_{2g+1} \geq 0$ will be introduced to eliminate the inequality sign of Eq. 16 and is used in Eq. 14 multiplied by a positive weight to replace ωΔIP. These changes lead to the equivalent problem of minimizing

$$G = \sum_{i=1}^{2g+1} \eta_i \, s_i \, , \quad \eta_i > 0 \quad (18)$$

subject to

$$\sum_{i=1}^{g} A_{ji} s_i - \sum_{i=1}^{g} A_{ji} s_{g+i} = \alpha M_j \, , \; j=1,2,..,m \quad (19)$$

$$\sum_{i=1}^{g} B_i s_i - \sum_{i=1}^{g} B_i s_{g+i} - s_{2g+1} = \gamma(/IP/+1) \quad (20)$$

$$s_k \geq 0 \, , \quad k = 1,2,.., 2g+1 \quad (21)$$

where

$$A_{ji} = \frac{\partial M_j}{\partial x_f} \frac{\partial x_f}{\partial a_i} + \frac{\partial M_j}{\partial a_i} \quad (22)$$

$$B_i = \frac{\partial IP}{\partial x_f} \frac{\partial x_f}{\partial a_i} + \frac{\partial IP}{\partial a_i} \quad (23)$$

4. NUMERICAL PROCEDURE

In this paper, a direct search numerical procedure is adopted. Thus, in each iteration, the increments in the vector of parameters a have to be determined such as to meet the requirement of getting closer to constraints satisfaction and to index of performance optimization. Using values of vector a from a previous iteration or from an initial guess, the values of matrix A, of vectors B and M, and of the scalar IP are calculated. From the solution of the problem defined by Eqs. 18-23, the variables s_k are obtained, leading to the determination of the vector of increments Δa, in Eq. 17, and to the definition of a new vector of parameters a. The process is repeated until the constraints are met with the desired accuracy and the oscillation in the values of IP are within limits compatible with this accuracy.

In the solution of the problem of Eqs. 18-23, α, γ and η_{2g+1} are parameters to be adjusted in each iteration in order to meet the search criterion objectives. It is important to notice that these objectives vary along the process of convergence. For example, when constraints are very near to be satisfied one can relax the objective of getting closer to meet the constraints in favor of the objective of decreasing IP.

The numerical procedure comprises typical phases in each iteration where numerical integration, numerical derivations and the solution of the linear programing problem are the principal ones. The computational characteristics concerning program simplicity and compactness, speed of convergence and reliability are very dependent upon the choices made in each of these phases. For the results presented here, the choices were the following: a low order Runge Kutta (2-4) with adjustable step size for the numerical integrations; the simplex algorithm with multipliers for the solution of the linear programing problem; and direct numerical derivations for obtaining the partial derivatives of state with respect to parameters.

The procedure adopted for obtaining the derivatives is specially important in terms of level of difficulty in computer implementation and processing time (Refs. 3-4). The choice of direct numerical derivation allows the organization of a computer program structured to be fitted for general application in the generation of numerical solutions of dynamical systems optimal control problems. Not having to recalculate, in each iteration, the partials of the state with respect to the parameters can save a significant amount of processing time, since the calculation of these partial derivatives is the most time consuming task. For the results presented here, recalculations were avoided by controlling the accumulated deviation in the vector of parameters since the last evaluation done.

5. PROCEDURE TEST

The problem selected to test the procedure was an Earth-Mars minimum time orbit transfer from a given initial circular orbit to a final circular orbit in the same plane, with fixed low thrust of controlled direction (Refs. 1,2,7). This problem can be formulated as the minimization of

$$IP = t_f \tag{24}$$

subject to the dynamical constraints

$$\dot{x}_1 = x_2 \tag{25}$$

$$\dot{x}_2 = x_3^2/x_1 - \mu/x_1^2 + T \operatorname{sen} \beta/(m_0 - \dot{m}t) \tag{26}$$

$$\dot{x}_3 = x_2 x_3/x_1 + T \cos \beta/(m_0 - \dot{m}t) \tag{27}$$

and to the boundary constraints, in normalized units (Ref. 7)

$$x_1(t_0) = 1.0;\ x_2(t_0) = 0.0;\ x_3(t_0) = 1.0;\ t_0 = 0;$$

$$x_1(t_f) = 1.5237;\ x_2(t_f) = 0.0;\ x_3(t_f) = 0.8101;$$

$$\mu = 1.0;\ m_0 = 1.0;\ \dot{m} = 0.074800391;$$

$$T = 0.14012969;$$

where, x_1 is the radial distance from spacecraft to Sun; x_2, the radial velocity; x_3, the tangential velocity; m, the mass of the spacecraft; μ, the gravitational constant of the attracting center; T, the thrust; and β, the control.

To simulate a situation of a state inequality constraint, in some of the tests an upper bound was taken for radial velocity ($\overline{x}_2 = 0.25$) such as to have

$$x_2(t) - \overline{x}_2 \leq 0 \tag{28}$$

An optimal solution was numerically generated, using an indirect optimal procedure for comparison and analysis in the evaluation of the quality of the suboptimal solutions obtained. The optimal control function (FOC) is shown in Figures 1-3, in correspondence with the optimal value for the index of performance, IPO (IPO = 3.31949).

Two situations were considered for the suboptimal approximation of the control function. In the first case, the interval $[t_0, t_f]$ was divided into a fixed number, (g-2), of subintervals, $[t_i, t_{i+1}]$, with $a_g = t_f$ and the values of control in the end points of these subintervals taken as the remaining (g-1) parameters to be optimized, and with the control $\beta(t)$, for $t_i \leq t \leq t_{i+1}$, given by

$$\beta(t) = a_j + (a_{j+1} - a_j)(t - t_i)(g-2)/a_g \tag{29}$$

where, $j = i+1$; $t_i = i\, a_g/(g-2)$, $i=0,1,..,g-2$.

In the second case, the control $\beta(t)$ was approximated by two arcs of parabola and the intermediate value of time, t_I, corresponding to the junction point, included among the parameters to be optimized. Under these conditions, for the first arc, it results that:

$$\beta(t) = a_1 + a_2 t_\omega + a_3 t_\omega^2 \tag{30}$$

where $t \leq t_I = a_7$ and $t_\omega = t/a_7$; and for the second arc, where $t > a_7$, $t_\omega = (t - a_7)/(a_8 - a_7)$ and $t_f = a_8$, there results

$$\beta(t) = a_4 + a_5 t_\omega + a_6 t_\omega^2 \tag{31}$$

To obtain the suboptimal control functions (SCF) for the test situations selected, the following cases, shown in Figures 1-4, were treated: (i) problem without state inequality constraints with the control approximation of Eqs. 30,31 (SCF1) and the approximation of Eq. 29, with g = 7 (SCF2) and g = 9 (SCF3); (ii) problem with state inequality constraint of Eq. 28 and with the control approximation of Eq. 29, taking g = 9 (SCF4) and g = 12 (SCF5). For an error of the order of 1.E-04 in the satisfaction of boundary constraints, the following results are shown in Table 1: convergence values of the parameters; number of evaluations of the matrix of the partial derivatives of final state with respect to the parameters (NE); percentual error (PE) of the suboptimal IP relative to the optimum value (IPO = 3.31949); and the relative processing time (RPT). In all cases tested, a straight line (defined by $t_0 = 0$, $\beta(t_0) = 0$, $t_f = 3.4$, $\beta(t_f) = 5$) was taken as the initial guess for control function (ICF). For the case with the two parabola arcs with optimized junction point, the initial guess for the intermediate time was $a_7 = 1.7$.

Table 1.

	SCF1	SCF2	SCF3	SCF4	SCF5
g	8	7	9	9	12
IP	3.323	3.325	3.322	3.548	3.516
PE	0.11%	0.16%	0.08%	-	-
a_1	.2868	.3650	.3458	1.038	.8243
a_2	1.435	.8095	.6680	1.163	.7664
a_3	0.000	1.064	.9325	.8770	1.013
a_4	4.163	4.873	1.336	2.975	.5335
a_5	2.383	5.196	4.709	-.4588	3.306
a_6	-1.170	5.471	5.040	5.670	2.564
a_7	1.670	3.325	5.261	5.129	-.3083
a_8	3.323	-	5.420	5.631	5.568
a_9	-	-	3.322	3.548	5.334
a_{10}	-	-	-	-	5.478
a_{11}	-	-	-	-	5.466
a_{12}	-	-	-	-	3.516
NE	12	16	14	11	16
RPT	1.0	1.3	1.4	2.0	3.8

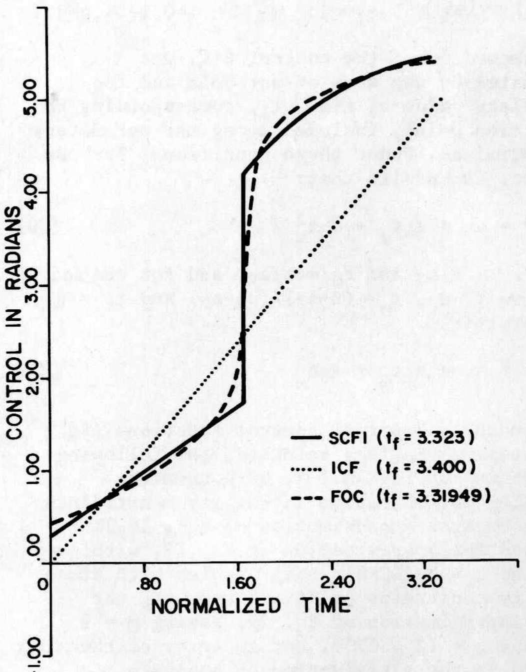

Figure 1. Two Parabola Arcs with Discontinuity

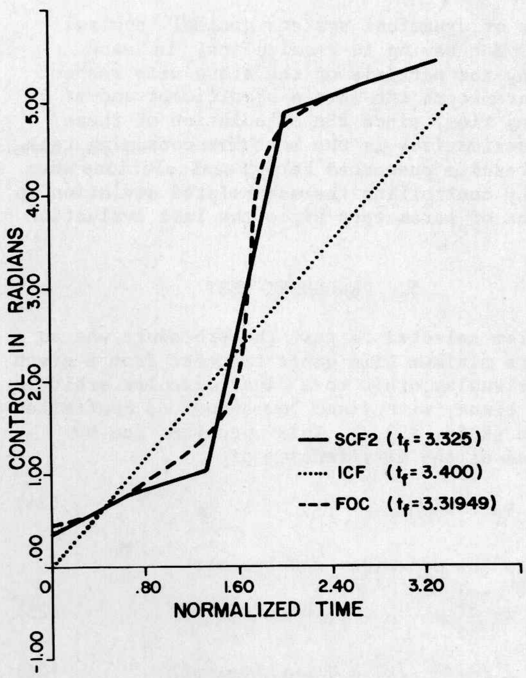

Figure 2. Linear Interpolations (g = 7)

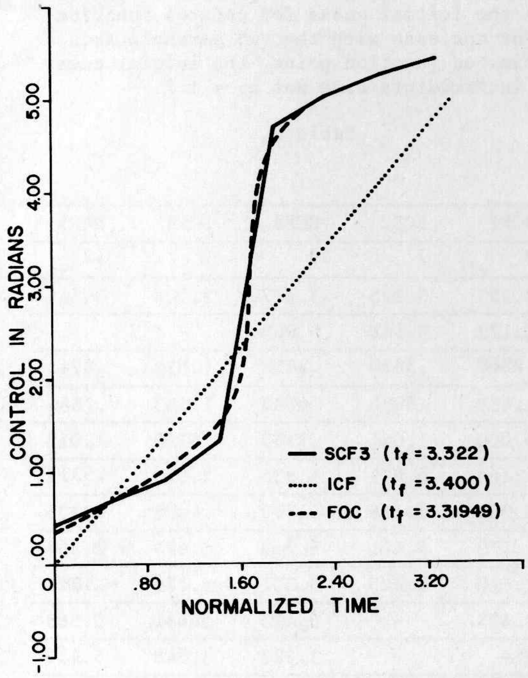

Figure 3. Linear Interpolations (g = 9)

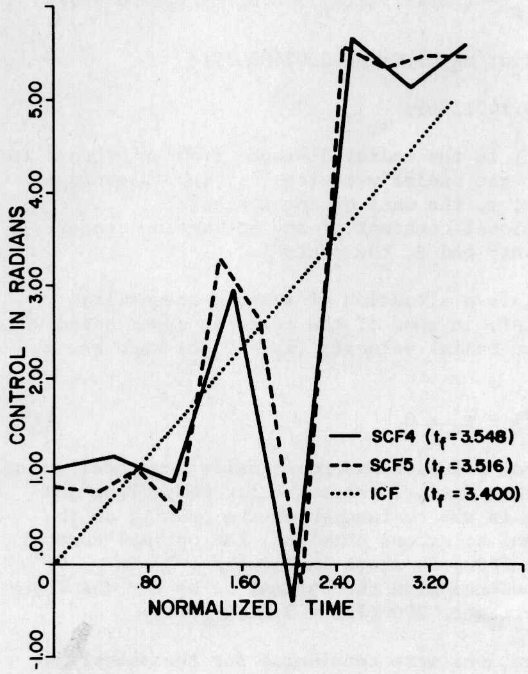

Figure 4. Linear Interpolations (g = 9, g = 12) with State Inequality Constraints

6. CONCLUSIONS

For most of the situations of practical interest, a numerical approach is necessary to get open loop optimal controls in the solution of dynamical systems control problems. The objective of simplifying numerical treatment justifies the adoption of suboptimal procedures of the type here presented. Under this aspect, the use of the Raileigh-Ritz method to obtain a control suboptimal approximation, together with the use of linear programing to solve the parameter optimization problem associated to each iteration, led to a procedure which has the characteristics of allowing: (i) simplicity of formulation and computer implementation; (ii) flexibility in the choice of the suboptimal control function form, including the possibility of choosing discontinuous solutions; (iii) possibility of naturally treating state inequality constraints; (iv) reduction in processing time and computer memory space as compared to optimal procedures; and (v) possibility of developing computer programs of general use in the solution of dynamical systems control problems.

The analysis of the results obtained in the numerical tests indicates that the previous advantageous features are not impaired by the quality of the suboptimal solutions given by the procedure. These solutions are very close to those corresponding to optimal procedures and are of perfectly acceptable accuracy for practical applications.

Finally, as could be expected, the adoption of an adequate approximate function form is important in terms of compatibility with the optimal control solution, control sensitivity with respect to parameter variations, and independence of the chosen parameters. A good choice seems to be that of taking the control given by linear interpolations, as indicated by the good quality of the results obtained in the test case. A significant advantage of this choice is to have a control function form which, in practice, is easy to implement.

7. REFERENCES

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